The Spectrum-Generating Algebra of the Van Hove Scenario is SO(8)

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The various nesting and pairing instabilities of the generalized Van Hove scenario can be classified via an $SO(8)$ spectrum-generating algebra. An $SO(6)$ subgroup is an approximate symmetry group of the model, having two 6-dimensional representations ('superspins'). This group contains as subgroups both the SO(5) and SO(4) groups found by Zhang, while one superspin is a combination of Zhang's 5-component superspin with a flux phase instability; the other includes a charge density wave instability plus s-wave superconductivity. This is the smallest group which can describe both striped phases and superconductivity.

Two groups play important roles in understanding a Hamiltonian: the *symmetry group* allows a classification of its degenerate eigenstates, while the Lie group of the *spectrum-generating algebra*[1] (SGA) can be used to analyze the complete spectrum. SGA's have proven to be useful in the study of collective modes in nuclear and high-energy physics, while in condensed matter physics they have been used to study phase transitions in liquid He and in one-dimensional (1D) metals[2]. In the 1D metals, the SGA is SU(8), with 63 elements and 56 possible order parameters including superconductivity and charge or spin density (CDW/SDW) waves. This algebra has also been applied to the two-dimensional (2D) Hubbard model[3]. However, we show that even for the generalized Hubbard model appropriate to the generalized Van Hove scenario, the appropriate SGA is SO(8), a considerably smaller algebra. This algebra has a natural subalgebra, SO(6), which acts as an approximate symmetry group generalizing Zhang[4], and including both his SO(5) and SO(4) as subgroups. We identify SO(6) as the smallest group which is capable of describing striped phases as well as superconductivity.

For a one-dimensional (1D) metal[5], nesting involves the two points of the Fermi surface, at $\pm k_F$, with k_F the Fermi momentum. Since this breaks momentum conservation (the states $+k_F$ and $-k_F$ become inequivalent), the *full group* SU(8) must be taken as the SGA[2]. On the other hand, in two dimensions, the dominant nesting arises at $\vec{Q} = (\pi, \pi)$, connecting the two Van Hove singularities (VHS's)[6] at $(\pi, 0)$ and $(0, \pi)$. Since the points $\pm(\pi, 0)$ are equivalent points of the reciprocal space lattice, nesting singularities involve only order parameters even in $\vec{k}$. Hence the SGA is a proper subgroup of SU(8) – the SO(8) algebra of Table I. (Note that there is some ambiguity in defining a SGA: here we define it as the algebra which contains the mean-field Hamiltonian.)

Δ_{s+}						
$i\Delta_{d-}$	τ					
$i\eta_-$	O_{CDW}	iO_{JC}				
Π_{x+}	$-iO_{JSx}$	O_{SDWx}	$-A_x$			
Π_{y+}	$-iO_{JSy}$	O_{SDWy}	$-A_y$	$-S_z$		
Π_{z+}	$-iO_{JSz}$	O_{SDWz}	$-A_z$	S_y	$-S_x$	
Q	$-i\Delta_{s-}$	Δ_{d+}	η_+	$-i\Pi_{x-}$	$-i\Pi_{y-}$	$-i\Pi_{z-}$

FIG. 1. Matrix Representation of SO(8), using the shorthand $O_\pm = O \pm O^\dagger$.

There is a combinatoric interpretation of this SO(8) which is independent of any particular hamiltonian. Consider an electronic system with a two-fold orbital degeneracy (labelled 1, 2) in addition to the spin degeneracy. The four creation operators can be written as $C^\dagger_{1\uparrow}$, $C^\dagger_{2\uparrow}$, $C^\dagger_{1\downarrow}$, and $C^\dagger_{2\downarrow}$. Including both particle-hole ($C^\dagger C$) and particle-particle ($C^\dagger C^\dagger$ or CC) operators, there are 28 pair operators, whose components define the Lie algebra of $SO(8)$ [recall that for $SO(N)$, the Lie algebra contains $N(N-1)/2$ elements]. Particular linear combinations of these elements are listed in Table I. Figure 1 rewrites these elements as an explicit representation of the Lie algebra of $SO(8)$. The 28 generators are the antisymmetric matrices L^{ij}, with matrix elements $L^{ij}_{kl} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k$. Figure 1 illustrates the equivalences as the lower half of an antisymmetric L-matrix. The operators satisfy the Lie algebra, with standard $SO(8)$ commutation rules

$$[L^{ij}, L^{km}] = i(\delta_{ik} L^{jm} + \delta_{jm} L^{ik} - \delta_{im} L^{jk} - \delta_{jk} L^{im}). \tag{1}$$

$SO(8-M)$ subalgebras can be formed by eliminating M rows of the L matrices, along with their corresponding columns. These will be designated as $\{I_1, ..., I_M\}$, where $I_1, ..., I_M$ are the rows (and columns) which have been eliminated. For instance, $\{234\}$ is the $SO(5)$ algebra studied by Zhang[4].

In a generalized Hubbard model[7], the creation operators can be expanded in terms of operators localized near the corresponding VHS's:

$$a^\dagger_{i\sigma} \simeq \frac{1}{2}\big((-1)^{x_i}\psi^\dagger_{1\sigma}(\vec{r}) + (-1)^{y_i}\psi^\dagger_{2\sigma}(\vec{r})\big), \tag{2}$$

where $\psi^\dagger_{1\sigma}$ and $\psi^\dagger_{2\sigma}$ are slowly varying functions of position $\vec{r} = a(x_i, y_i)$. A more precise definition is given

Table I: Generators of SO(8) Lie Algebra

Operator	Representation
Q	$(C_{1\uparrow}^\dagger C_{1\uparrow} + C_{2\uparrow}^\dagger C_{2\uparrow} + C_{1\downarrow}^\dagger C_{1\downarrow} + C_{2\downarrow}^\dagger C_{2\downarrow})/2 - 1$
τ	$(C_{1\uparrow}^\dagger C_{1\uparrow} - C_{2\uparrow}^\dagger C_{2\uparrow} + C_{1\downarrow}^\dagger C_{1\downarrow} - C_{2\downarrow}^\dagger C_{2\downarrow})/2$
S_z	$(C_{1\uparrow}^\dagger C_{1\uparrow} + C_{2\uparrow}^\dagger C_{2\uparrow} - C_{1\downarrow}^\dagger C_{1\downarrow} - C_{2\downarrow}^\dagger C_{2\downarrow})/2$
A_z	$(C_{1\uparrow}^\dagger C_{1\uparrow} - C_{2\uparrow}^\dagger C_{2\uparrow} - C_{1\downarrow}^\dagger C_{1\downarrow} + C_{2\downarrow}^\dagger C_{2\downarrow})/2$
S_x	$(C_{1\uparrow}^\dagger C_{1\downarrow} + C_{2\uparrow}^\dagger C_{2\downarrow} + C_{1\downarrow}^\dagger C_{1\uparrow} + C_{2\downarrow}^\dagger C_{2\uparrow})/2$
A_x	$(C_{1\uparrow}^\dagger C_{1\downarrow} - C_{2\uparrow}^\dagger C_{2\downarrow} + C_{1\downarrow}^\dagger C_{1\uparrow} - C_{2\downarrow}^\dagger C_{2\uparrow})/2$
iS_y	$(C_{1\uparrow}^\dagger C_{1\downarrow} + C_{2\uparrow}^\dagger C_{2\downarrow} - C_{1\downarrow}^\dagger C_{1\uparrow} - C_{2\downarrow}^\dagger C_{2\uparrow})/2$
iA_y	$(C_{1\uparrow}^\dagger C_{1\downarrow} - C_{2\uparrow}^\dagger C_{2\downarrow} - C_{1\downarrow}^\dagger C_{1\uparrow} + C_{2\downarrow}^\dagger C_{2\uparrow})/2$
O_{CDW}	$(C_{1\uparrow}^\dagger C_{2\uparrow} + C_{2\uparrow}^\dagger C_{1\uparrow} + C_{1\downarrow}^\dagger C_{2\downarrow} + C_{2\downarrow}^\dagger C_{1\downarrow})/2$
O_{SDWz}	$(C_{1\uparrow}^\dagger C_{2\uparrow} + C_{2\uparrow}^\dagger C_{1\uparrow} - C_{1\downarrow}^\dagger C_{2\downarrow} - C_{2\downarrow}^\dagger C_{1\downarrow})/2$
O_{JC}	$(C_{1\uparrow}^\dagger C_{2\uparrow} - C_{2\uparrow}^\dagger C_{1\uparrow} + C_{1\downarrow}^\dagger C_{2\downarrow} - C_{2\downarrow}^\dagger C_{1\downarrow})/2$
O_{JSz}	$(C_{1\uparrow}^\dagger C_{2\uparrow} - C_{2\uparrow}^\dagger C_{1\uparrow} - C_{1\downarrow}^\dagger C_{2\downarrow} + C_{2\downarrow}^\dagger C_{1\downarrow})/2$
O_{SDWx}	$(C_{1\uparrow}^\dagger C_{2\downarrow} + C_{2\uparrow}^\dagger C_{1\downarrow} + C_{1\downarrow}^\dagger C_{2\uparrow} + C_{2\downarrow}^\dagger C_{1\uparrow})/2$
iO_{SDWy}	$(C_{1\uparrow}^\dagger C_{2\downarrow} + C_{2\uparrow}^\dagger C_{1\downarrow} - C_{1\downarrow}^\dagger C_{2\uparrow} - C_{2\downarrow}^\dagger C_{1\uparrow})/2$
O_{JSx}	$(C_{1\uparrow}^\dagger C_{2\downarrow} - C_{2\uparrow}^\dagger C_{1\downarrow} + C_{1\downarrow}^\dagger C_{2\uparrow} - C_{2\downarrow}^\dagger C_{1\uparrow})/2$
iO_{JSy}	$(C_{1\uparrow}^\dagger C_{2\downarrow} - C_{2\uparrow}^\dagger C_{1\downarrow} - C_{1\downarrow}^\dagger C_{2\uparrow} + C_{2\downarrow}^\dagger C_{1\uparrow})/2$

Op.	Representation	Op.	Representation
Δ_s	$(C_{1\uparrow}C_{1\downarrow} + C_{2\uparrow}C_{2\downarrow})/2$	$\Delta_s^\dagger$	$(C_{1\downarrow}^\dagger C_{1\uparrow}^\dagger + C_{2\downarrow}^\dagger C_{2\uparrow}^\dagger)/2$
Δ_d	$(C_{1\uparrow}C_{1\downarrow} - C_{2\uparrow}C_{2\downarrow})/2$	$\Delta_d^\dagger$	$(C_{1\downarrow}^\dagger C_{1\uparrow}^\dagger - C_{2\downarrow}^\dagger C_{2\uparrow}^\dagger)/2$
$-i\Pi_y$	$(C_{2\uparrow}C_{1\uparrow} + C_{2\downarrow}C_{1\downarrow})/2$	$i\Pi_y^\dagger$	$(C_{1\uparrow}^\dagger C_{2\uparrow}^\dagger + C_{1\downarrow}^\dagger C_{2\downarrow}^\dagger)/2$
Π_x	$(C_{2\uparrow}C_{1\uparrow} - C_{2\downarrow}C_{1\downarrow})/2$	$\Pi_x^\dagger$	$(C_{1\uparrow}^\dagger C_{2\uparrow}^\dagger - C_{1\downarrow}^\dagger C_{2\downarrow}^\dagger)/2$
η	$(C_{1\uparrow}C_{2\downarrow} + C_{2\uparrow}C_{1\downarrow})/2$	$\eta^\dagger$	$(C_{2\downarrow}^\dagger C_{1\uparrow}^\dagger + C_{1\downarrow}^\dagger C_{2\uparrow}^\dagger)/2$
Π_z	$(C_{1\uparrow}C_{2\downarrow} - C_{2\uparrow}C_{1\downarrow})/2$	$\Pi_z^\dagger$	$(C_{2\downarrow}^\dagger C_{1\uparrow}^\dagger - C_{1\downarrow}^\dagger C_{2\uparrow}^\dagger)/2$

Table II: Interaction Terms

$(G_2 - G_3)\Sigma_{\vec r}\Delta_d^\dagger(\vec r)\Delta_d(\vec r)$
$(G_2 + G_3)\Sigma_{\vec r}\Delta_s^\dagger(\vec r)\Delta_s(\vec r)$
$(2G_1 + G_3 - G_4)\Sigma_{\vec r}[O_{CDW}(\vec r)]^2$
$(G_3 + G_4 - 2G_1)\Sigma_{\vec r}[O_{JC}(\vec r)]^2$
$(G_4 - G_3)\Sigma_{\vec r}\vec O_{JS}(\vec r)\cdot\vec O_{JS}(\vec r)$
$-(G_3 + G_4)\Sigma_{\vec r}\vec O_{SDW}(\vec r)\cdot\vec O_{SDW}(\vec r)$

in Refs.[8,9]. The Lie algebra of Table I corresponds to $O \to \Sigma_{\vec r} O(\vec r)$, with $C_{i\sigma}^\dagger \to \psi_{i\sigma}^\dagger(\vec r)$, etc. With this definition, the operators become equivalent to those introduced by Schulz[7] and Zhang[4]. The SGA $\mathbf{G}$ is defined in Fourier space as $\mathbf{G} = \oplus_{\vec k}\mathbf{g}_{\vec k}$, where $\mathbf{g}_{\vec k}$ is the algebra of a particular $\vec k$-component of the Fourier transformed operators of Table I.

The interaction terms in the generalized Hubbard hamiltonian[7] can be written in terms of pairs of these operators, Table II. Here the G_i's are coupling constants, which can be related to the Hubbard U and to various near-neighbor interaction terms[7]. For the pure Hubbard model, $G_1 = G_2 = G_3 = G_4 = U/4\pi t$ and t is the nearest neighbor hopping parameter. The form of the interaction term is not unique, since a number of alternative terms can arise by anticommuting the operators.

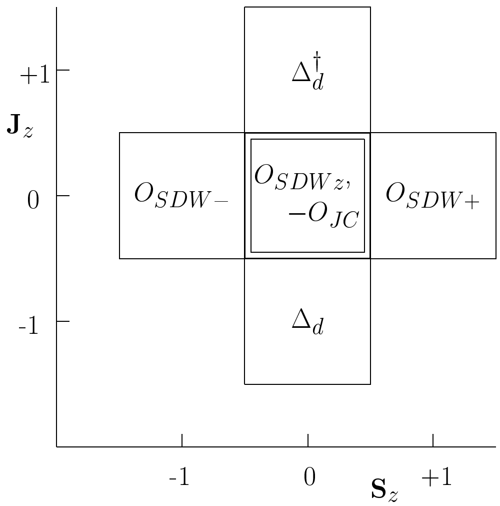

FIG. 2. $SO(4)$ weight diagram of $\mathbf{V}'$, where S_z (J_z) is the z component of spin (pseudospin).

While SO(8) is the SGA of the generalized Hubbard model, Zhang's SO(5) is an (approximate) symmetry algebra of the same model – in particular, the collective modes (SDW and d-wave superconductivity for SO(5)) are *elements* of the SGA, but are *components* of a superspin, which transforms under the symmetry group. For the generalized Hubbard model, the natural (approximate) symmetry group is SO(6), defined as follows. $SO(8)$ contains an $SO(3)$ algebra, which we call *isospin*, generated by $T_0 = \tau$, $T_\pm = O_{CDW} \pm O_{JC}$. This algebra is the algebra of the VHS's: the z-component of the isospin, T_0, measures the excess population of the 1-VHS over the 2-VHS. Those operators which do not commute with T_0 can lead to a nesting or pairing instability. Hence, the important transformation group of the VHS's is the $SO(6)$ subgroup $\{23\}$ which commutes with T_0, leading to the decomposition scheme:

$$\mathbf{SO(8)} \to \mathbf{SO(6)}_{\{23\}} \oplus \mathbf{V}_+ \oplus \mathbf{V}_- \oplus \tau, \qquad (3)$$

under which $\mathbf{28} \to (\mathbf{15},0)+(\mathbf{6},1)+(\mathbf{6},-1)+(\mathbf{1},0)$, where $(\mathbf{m}, n)$ denotes representation $\mathbf{m}$ of $SO(6)$ and eigenvalue n of T_0. The 6-vectors can be denoted $\mathbf{V}_\pm = \mathbf{V} \pm \mathbf{V}'$ with

$$\mathbf{V} = \{L_{21}, L_{42}, L_{52}, L_{62}, L_{72}, L_{82}\}$$
$$\mathbf{V}' = \{L_{31}, L_{43}, L_{53}, L_{63}, L_{73}, L_{83}\}, \qquad (4)$$

shown boxed in Fig. 1. The group structure of $\mathbf{V}'$ is shown in Fig. 2, where J_z is the z-component of the pseudospin operator introduced by Yang and Zhang[10] and $O_{SDW\pm} = \mp(O_{SDWx} \pm iO_{SDWy})/\sqrt{2}$. An analogous diagram can be drawn for $\mathbf{V}$. The group $SO(6)_{\{23\}}$ transforms the components of each of these 6-vectors among themselves, without mixing the two vectors, while τ transforms the vectors into each other.

A number of points should be noted. (1) The $SO(6)$ group {23} contains Zhang's $SO(5)$ group as a subgroup, as well as the the $SO(4)$ group introduced by Yang and Zhang[10]. Moreover, $\mathbf{V}'$, Fig. 2, combines Zhang's $SO(5)$ superspin with O_{JC}, which is essentially equivalent to the flux phase[11].

(2) The twelve components of superspin are precisely the collective modes identified earlier by Schulz[7], and most of them have been found to play an important role in the cuprates: s-wave superconductivity in electron-doped and (possibly) overdoped cuprates, CDW's near optimal doping[12,13], the flux phase near half filling[14,13].

(3) For a bare band dispersion (neglecting interactions) of the form

$$\epsilon_{\vec{k}} = -2t(\cos k_x a + \cos k_y a) + 4t' \cos k_x a \cos k_y a, \quad (5)$$

two parameters control the symmetry of the quadratic part of the generalized Hubbard hamiltonian, t' and $\tilde{\mu} = E_F - E_V$, the shift of the Fermi level E_F from the VHS E_V. When both parameters are zero (half filling with square Fermi surface) the hamiltonian has an extra pseudospin symmetry[10]. In this case, the nature of the ground state instability is controlled solely by the interaction terms (the G's). A pure Hubbard interaction (U) breaks the $SO(6)$ symmetry (Table II):

$$\mathbf{SO(6)} \rightarrow \mathbf{SO(3)} \oplus \mathbf{SO(3)}, \quad (6)$$

with one $SO(3)$ ordinary spin, and the other the pseudospin[10]. Both 6-vectors are broken down to pairs of 3-vectors

$$\mathbf{V} \rightarrow \{\mathbf{O_{JSi}}\} \oplus \{\mathbf{\Delta_s^\dagger}, \mathbf{\Delta_s}, \mathbf{O_{CDW}}\}$$
$$\mathbf{V}' \rightarrow \{\mathbf{O_{SDWi}}\} \oplus \{\mathbf{\Delta_d^\dagger}, \mathbf{\Delta_d}, \mathbf{O_{JC}}\}; \quad (7)$$

however, there remains an accidental degeneracy of one vector (O_{JS}) with the opposite pseudovector. At half filling the lowest energy state is O_{SDW}. As discussed below, this weak coupling result must be corrected for strong correlation effects.

When the Fermi surface is distorted from square, either by doping away from half filling ($\tilde{\mu}$) or by introducing second-neighbor hopping terms t', the pseudospin degeneracy is broken, in such a way as to *favor pairing over nesting instabilities*. This can be seen by Hartree-Fock[15] or renormalization group[16,17,12] analyses or by a linear response analysis (following Ref.[18]).

If the superspin is written as $\vec{O}$ (a twelve component vector incorporating both representations), then in linear response theory it is assumed that there is an applied field $\vec{h}_O$ (also a 12-vector) which couples to $\vec{O}$. The hamiltonian in the presence of $\vec{h}_O$ is

$$H = \Sigma_{\vec{k}\sigma}\epsilon_{\vec{k}} a^\dagger_{\vec{k}\sigma} a_{\vec{k}\sigma} + \vec{h}_O \cdot \vec{O} \quad (8)$$

(here the terms in G have been neglected), with resulting free energy

$$F_0(\vec{O}) = \Omega_0(\mu, \vec{O}, T) + \mu N$$
$$\Omega_0(\mu, \vec{O}, T) = -2k_B T \Sigma_{\vec{k}\sigma} \ln(1 + e^{-(E_{\vec{k}\sigma} - \mu)/k_B T}) + \vec{h}_O \cdot \vec{O}, \quad (9)$$

with $E_{\vec{k}\sigma}$ the quasiparticle energy found by applying a Bogoliubov-Valentin transformation to Eq. 8 or via SGA techniques[1,2]. The expectation value of each superspin component O_i is found from

$$\frac{\partial \Omega_0}{\partial h_{Oi}} = 0, \quad (10)$$

and the corresponding susceptibility is

$$\chi_{0i} = \lim_{\vec{h}_O \rightarrow 0} \left(\frac{O_i}{h_{Oi}}\right). \quad (11)$$

Including the interaction terms, the Hartree-Fock free energy becomes

$$F_{HF}(\vec{O}) = \Sigma_i \Big(\frac{1}{2\chi_{0i}} + G_i\Big) O_i^2, \quad (12)$$

leading to an instability of the ith mode when

$$1 + 2\chi_{0i} G_i = 0. \quad (13)$$

If the quadratic hamiltonian is symmetric under $SO(6)$, then the component with the most negative G_i is the first to diverge. For finite $\tilde{\mu}$ or t',the hamiltonian still preserves particle number, so there are only two independent susceptibilities, the particle-hole susceptibility χ_{00} and the pair susceptibility χ_{02}, with

$$\chi_{00} = -2\Sigma_{\vec{k}\sigma} \frac{f(\epsilon_{\vec{k}\sigma})}{\epsilon_{\vec{k}\sigma} - \epsilon_{\vec{k}+\vec{Q},\sigma}}$$
$$\chi_{02} = -\Sigma_{\vec{k}\sigma} \frac{f(\epsilon_{\vec{k}\sigma})}{\epsilon_{\vec{k}\sigma} - \epsilon_F}. \quad (14)$$

Note that in nearest-neighbor hopping models ($t' = 0$) $\epsilon_{\vec{k}+\vec{Q},\sigma} = -\epsilon_{\vec{k}\sigma}$, and the two expressions become equivalent when $\epsilon_F = 0$ – i.e., at half filling.

Figure 3 illustrates the doping dependence of these susceptibilities for a Hubbard band with nearest-neighbor hopping only ($t' = 0$). The point of maximum instability (largest χ) coincides with the point at which the VHS

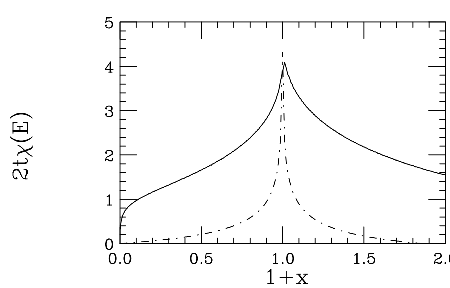

FIG. 3. Susceptibilities χ_{00} (dotdashed line) and χ_{02} (solid line) vs. band filling $1 + x$ for Eq. 5 with $t' = 0$.

crosses the Fermi level – half filling when $t' = 0$. When $x = t' = 0$, the susceptibilities are degenerate, $\chi_{00} = \chi_{02}$, as expected from the pseudospin symmetry[10]. However, as soon as the system is doped away from half filling ($x \neq 0$) the electron-hole susceptibility drops precipitously, whereas the pair susceptibility falls off much more gradually. A similar effect arises if the system is maintained at optimal doping (the VHS), but the parameter t' is varied – indeed χ_{02} actually increases with increasing t', Fig 4. This striking difference is readily understood: the electron-hole susceptibility involves inter-VHS nesting, which gets progressively worse as the Fermi surface gets more curved, whereas the electron-electron susceptibility involves intra-VHS scattering, and increases with t' as the Fermi surfaces become nearly 1D near the VHS's. (In Figs. 3-4, the logarithmic divergence at the VHS was cut off by adding a small imaginary term to the denominator of χ.)

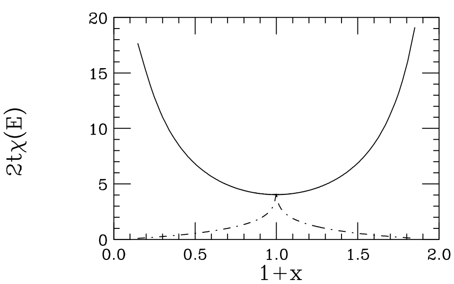

FIG. 4. As in Fig. 3, but with $4t' = E_F$.

Figure 3 is consistent with the RG results of Schulz[7]. From Eq. 13, when all χ's are equal the order parameter associated with the most negative G is the first to go singular. For the pure Hubbard model, this means that the leading instability at half filling is the SDW. When the material is doped, the SDW susceptibility plummets, and at some point d-wave superconductivity becomes favorable. In agreement with Zhang[4], the shift of the Fermi energy from the VHS is a relevant parameter in driving this SDW→ d-wave superconducting transition.

Despite the simplicity of this picture, a purely SO(5) model cannot explain the full physics of the cuprates. First, the above analysis is in the weak coupling limit, and a strong coupling reanalysis of Table II (J-term dominant) shows that the flux phase – not included in SO(5) – is the lowest energy state. A second problem is in $La_{2-x}Ba_xCuO_4$ and $La_{2-x-y}Nd_ySr_xCuO_4$, near $x = 1/8$, where the striped phase is commensurately pinned, leading to long-ranged magnetic and charge order[19]. At the same time, superconductivity is strongly suppressed, demonstrating that whatever the driving force for charge order may be, it is not superconductivity, but is in competition with superconductivity. Since SO(5) only allows for antiferromagnetism and superconductivity, it does not have sufficient flexibility to properly describe this situation. There are strong hints that the charged stripes are associated with a CDW: the low-temperature tetragonal phase is nearly coterminus with the long-range SDW-ordered phase, and the fact that the charged stripes are best seen by neutron diffraction suggests a strong associated lattice distortion. There is considerable additional evidence that phonons and structural instabilities play an important role in the doped material[12]. Hence, for a detailed description of the doping dependence of the pseudogap, striped phases, and extended VHS's, it may be necessary to recognize that strong electron-phonon coupling can lead to a crossover to a groundstate involving the **V** 6-vector[13].

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